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Parameterization of the Autoconversion Process. Part I: Analytical Formulation of the Kessler-Type Parameterizations

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ABSTRACT

Various commonly used Kessler-type parameterizations of the autoconversion of cloud droplets to embryonic raindrops are theoretically derived from the same formalism by applying the generalized mean value theorem for integrals to the general collection equation. The new formalism clearly reveals the approximations and assumptions that are implicitly embedded in these different parameterizations. A new Kessler-type parameterization is further derived by eliminating the incorrect and/or unnecessary assumptions inherent in the existing Kessler-type parameterizations. The new parameterization exhibits a different dependence on liquid water content and droplet concentration, and provides theoretical explanations for the multitude of values assigned to the tunable coefficients associated with the commonly used parameterizations. Relative dispersion of the cloud droplet size distribution (defined as the ratio of the standard deviation to the mean radius of the cloud droplet size distribution) is explicitly included in the new parameterization, allowing for investigation of the influences of the relative dispersion on the autoconversion rate and, hence, on the second indirect aerosol effect. The new analytical parameterization compares favorably with those parameterizations empirically obtained by curve-fitting results from simulations of detailed microphysical models.

1. Introduction

Rain is initiated in liquid water clouds by collision and coalescence of cloud droplets wherein larger droplets with higher settling velocities collect smaller droplets and become embryonic raindrops. This so-called autoconversion process is usually the dominant process that leads to the formation of drizzle in stratiform clouds. Accurate parameterization of the autoconversion process in atmospheric models of various scales [from large eddy simulations (LESs) to global climate models] is important for understanding the interactions between cloud microphysics and cloud dynamics (e.g., Chen and Cotton 1987), for the forecasting of the freezing drizzle formation and aircraft icing (Rasmussen et al. 2002), and for improving the treatment of clouds in climate models (Rotstayn 2000).

Kessler (1969) proposed a simple parameterization that linearly relates the autoconversion rate to the cloud liquid water content, and this parameterization has been widely used in cloud-related modeling studies because of its simplicity. But this simple parameterization leaves much to be desired, as it is well known that the autoconversion rate is a function of not only of the liquid water content but also the cloud droplet number concentration and the spectral shape of the cloud droplet size distribution (Berry 1967; Berry and Reinhardt 1974a,b). Over the last several decades, much effort has been devoted to improving the original Kessler parameterization by including the effect of the droplet concentration as well as liquid water content (Manton and Cotton 1977; Tripoli and Cotton 1980; Liou and Ou 1989; Baker 1993). The effort to improve the parameterization of the autoconversion rate has been recently reinforced by an increasing interest in cloud–climate interactions, and particularly in studies of the second indirect aerosol effect (Boucher et al. 1995; Lohmann and Fleichter 1997; Rotstayn 2000).

Without loss of generality, all of the Kessler-type parameterizations can be written as

$$P = cLH(y - y_c), \tag{1}$$

where P is the autoconversion rate (in g cm^{-3} s^{-1}), c is an empirical coefficient (in s^{-1}) (hereafter conversion coefficient), and L is the cloud liquid water content (in g cm^{-3}). The Heaviside step function $H(y - y_c)$ is introduced to describe a threshold y_c (hereafter threshold coefficient) below which the autoconversion is negligibly small. The meaning of y is different in different parameterizations; for example, y represents the cloud liquid water content in the original Kessler parameterization, whereas it represents the mean volume radius in the Manton and Cotton expression, and the mean

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radius of the fourth moment in the parameterizations of Liou and Ou (1989), Baker (1993), and Boucher et al (1995). It is noteworthy that, while the autoconversion rate is also formulated in terms of the cloud water mixing ratio instead of the liquid water content, transformation between these two equivalent formulations is straightforward.

A common problem with the Kessler-type parameterizations is that they collectively lack a solid theoretical foundation, approximations associated with their use are not clear, and the logical connections between the various Kessler-type parameterizations are not well understood. Here we first derive the various existing Kessler-type parameterizations by applying the generalized mean value theorem to the general collection equation. This derivation readily reveals the distinctions between, and approximations of, these different parameterizations. The existing Kessler-type parameterizations are then generalized into a unified expression that includes the effect of the spectral shape of the cloud droplet size distribution as well as the droplet concentration and liquid water content. A new Kessler-type parameterization that eliminates the incorrect and/or unnecessary assumptions inherent in the existing parameterizations is further developed and applied to explain the multitude of the empirical coefficients associated with the existing Kessler-type parameterizations. The effect of the spectral shape on the autoconversion rate is also discussed using the new parameterization.

2. Reexamination of typical Kessler-type parameterizations

As discussed above, one of the problems shared by the existing Kessler-type parameterizations is the lack of a rigorous theoretical basis for their formulation. The purpose of this section is to show that application of the generalized mean value theorem for integrals to the general equation for the autoconversion rate provides the required theory.

a. Autoconversion rate and generalized mean value theorem for integrals

We first recapitulate the expression for the autoconversion rate and the generalized mean value theorem for integrals that will be used in this work. From the continuous collection equation, the mass growth rate of a collector drop of radius R falling through a population of smaller droplets having a cloud droplet size distribution $n(r)$ is given by (Pruppacher and Klett 1997)

$$\frac{dm(R)}{dt} = \int K(R, r)m(r)n(r)\, dr. \tag{2}$$

The autoconversion rate P is obtained by further integrating (2) over all collector drops:

$$P = \int \frac{dm}{dt} n(R)\, dR = \int n(R)\, dR \int K(R, r)m(r)n(r)\, dr. \tag{3}$$

The interval of the integration is from the smallest to the largest droplets, and is omitted throughout the paper for simplicity. It is well known from standard calculus textbooks (e.g., Spiegel 1992) that, if $f(x)$ and $g(x)$ are continuous in the interval $x \in [a, b]$ and $g(x)$ does not change sign in this interval, then there is a point $x_\xi \in (a, b)$ such that

$$\int_a^b f(x)g(x)\, dx = f(x_\xi) \int_a^b g(x)\, dx. \tag{4}$$

It will be shown below that application of the generalized mean value theorem for integrals to (3) provides a unified basis for the various Kessler-type parameterizations.

b. Derivation of the typical Kessler-type parameterizations

Kessler (1969) intuitively proposed an expression for the autoconversion rate such that

$$P_K = a_K(L - L_c), \tag{5a}$$

where L_c is the threshold liquid water content below which the autoconversion rate is assumed to be so small that the empirical coefficient $a_K = 0$ when $L \le L_c$, and $a_K > 0$ when $L > L_c$. This parameterization can be rewritten as (Khairoutdinov and Kogan 2000)

$$P_K = a_K(L - L_c)H(L - L_c), \tag{5b}$$

where the Heaviside function is introduced to represent the threshold process. Equation (5b) can be further expressed in the form of (1):

$$P_K = c_K LH(L - L_c). \tag{5c}$$

Comparison of (5c) with (5b) yields

$$c_K = a_K\left(1 - \frac{L_c}{L}\right), \tag{5d}$$

Equation (5d) provides an explanation for the results obtained by Kessler (1969) that increasing a_K affects precipitation development in much the same way as does decreasing the threshold liquid water content because the autoconversion rate increases when a_K increases or the threshold liquid water content decreases for a given liquid water content.

The Kessler parameterization can also be derived by applying the generalized mean value theorem for integrals to (3) as follows. Application of the generalized mean value theorem to the first integral of (3) yields

$$P = \int K(R, r_K)n(R)\,dR \int m(r)n(r)\,dr$$
$$= L \int K(R, r_K)n(R)\,dR, \tag{6}$$

where r_K is between the smallest and the largest cloud droplets. Further application of the generalized mean value theorem to (6) yields

$$P = K(R_K, r_K)NL = c_K L, \tag{7}$$

where R_K is between the smallest and the largest cloud droplets, N is the total number concentration of cloud droplets, and $K(R_K, r_K)$ represents the "average" collection kernel. Equation (7) becomes the Kessler parameterization if the conversion rate satisfies (5d). The above derivation shows that the original Kessler parameterization with constant values of a_K and L_c results from the assumption of a fixed collection kernel that is independent of droplet radius and proportional to $(1 - L_c/L)N^{-1}$. These assumptions are clearly not valid.

Manton and Cotton (1977; see also Tripoli and Cotton 1980) formulated a similar expression for the autoconversion rate

$$P_{\mathrm{MC}} = c_{\mathrm{MC}} L H(L - L_c). \tag{8}$$

Unlike the original Kessler parameterization, however, the conversion coefficient was further expressed as

$$c_{\mathrm{MC}} = \pi E_{\mathrm{MC}} R_3^2 V(R_3) N, \tag{9}$$

where E_{MC} represents an average collection efficiency associated with the autoconversion process, R_3 is the mean volume radius (see the appendix for the definition of R_3), and $V(R_3)$ is the terminal velocity of a droplet of radius R_3. They also argued that the threshold of the autoconversion process was determined by the value of the mean volume radius instead of by the liquid water contents such that

$$L_c = \frac{4\pi\rho_w}{3} R_{3c}^3 N, \tag{10}$$

where R_{3c} is the threshold mean volume radius, and ρ_w is the density of water. Manton and Cotton used $E_{\mathrm{MC}} = 0.55$, and $R_{3c} = 10\ \mu\mathrm{m}$.

The Manton–Cotton parameterization can also be derived by applying the mean value theorem to the collection equation, but in a slightly different way than for the original Kessler parameterization. The collection kernel $K(R, r)$ depends generally on the collection efficiency E and the terminal velocity V and is given by

$$K(R, r) = E(R, r)\pi(R + r)^2[V(R) - V(r)]. \tag{11}$$

Because cloud droplets are so small, this equation can be simplified by assuming that

$$(R + r)^2 \approx R^2 \quad \text{and} \tag{12a}$$

$$V(R) - V(r) \approx V(R). \tag{12b}$$

Substitution of (12a), (12b), and (11) into (3) yields

$$P = \pi \int R^2 V(R)n(R)\,dR \int E(R, r)m(r)n(r)\,dr. \tag{13}$$

Application of the generalized mean-value theorem to the first integral of (13) yields

$$P = \pi L \int R^2 V(R)n(R)E(R, r_{\mathrm{MC}})\,dR. \tag{14}$$

Further application of the generalized mean-value theorem to (14) yields

$$P = \pi L R_{\mathrm{MC}}^2 V(R_{\mathrm{MC}}) E_{\mathrm{MC}} \int n(R)\,dR$$
$$= \pi E_{\mathrm{MC}} R_{\mathrm{MC}}^2 V(R_{\mathrm{MC}}) NL. \tag{15}$$

Comparison of (15) to (8) and (9) shows that (15) reduces to the Manton–Cotton parameterization under the assumption of $R_{\mathrm{MC}} = R_3$. This assumption is invalid except in the case of a monodisperse cloud droplet size distribution.

The familiar form of the Manton–Cotton parameterization can be derived by assuming that the terminal velocity of the drop R is well described by the Stokes law

$$V(R) = \kappa_1 R^2, \tag{16}$$

where $\kappa_1 = 1.19 \times 10^6\ \mathrm{cm^{-1}\ s^{-1}}$ is the Stokes constant. Substitution of (16) into (15) yields

$$R_{\mathrm{MC}} = \pi\kappa_1 E_{\mathrm{MC}} R_3^4 NL. \tag{17a}$$

Substitution of the expression relating the mean volume radius to the liquid water content and droplet concentration into (17a) yields the familiar form of the Manton–Cotton parameterization:

$$P_{\mathrm{MC}} = \alpha_{\mathrm{MC}} N^{-1/3} L^{7/3} H(R_3 - R_{3c}), \tag{17b}$$

where the parameter

$$\alpha_{\mathrm{MC}} = \pi\kappa_1 \left(\frac{3}{4\pi\rho_w}\right)^{4/3} E_{\mathrm{MC}}. \tag{17c}$$

The Heaviside function $H(R_3 - R_{3c})$ is introduced to consider the threshold process such that the autoconversion rate is negligibly small when $R_3 < R_{3c}$.

A major improvement of the Manton–Cotton parameterization over the original Kessler parameterization is inclusion of the droplet concentration as a dependent variable in formulation of the autoconversion rate, which enables one to differentiate between airmass types. Another improvement is that the threshold is determined by the mean volume radius rather than the liquid water content; this change makes physical sense because a cloud with a large liquid water content, a large number of droplets, and therefore a small mean volume radius will not rain. It is evident from the above derivation that these improvements result from relaxing the assumption of a fixed collection kernel (independent of the droplet radius) inherent in the original Kessler parameterization. The derivation also exposes the following deficiencies remaining in the Manton–Cotton parameterization: fixed collection efficiency, terminal velocity, and $R_{MC} = R_3$.

Several parameterizations that are slightly different from the Manton–Cotton parameterization have been subsequently proposed. Instead of applying the mean-value theorem to the integral of (14) before substituting the Stokes law for the terminal velocity, Liou and Ou (1989) relaxed the assumption of a fixed terminal velocity by first applying the Stokes law for the terminal velocity and obtained the autoconversion rate

$$P = \pi\kappa_1 L \int E(R, r_{MC}) R^4 n(R)\, dR. \tag{18}$$

Application of the generalized mean-value theorem to (18) yields

$$P_{LO} = \pi\kappa_1 E_4 L \int R^4 n(R)\, dR = \pi\kappa_1 E_4 R_4^4 NL, \tag{19}$$

where E_4 is the average collection efficiency associated with (18) and R_4 is the mean radius of the fourth moment (see the appendix for the definition of R_4). Liou and Ou assumed a fixed linear relation between R_4 and the mean-square radius R_2, $R_4 = 1.247R_2$, and investigated sensitivities of cloud radiative properties to the mean-square radius.

In investigation of the behavior of cloud condensation nuclei in the marine cloud-topped boundary layer, Baker (1993) used a similar parameterization but assumed R_4 is equal to the mean volume radius R_3 such that

$$P_{Baker} = \pi\kappa_1 \left(\frac{3}{4\pi\rho_w}\right)^{4/3} E_4 \gamma N^{-1/3} L^{7/3} H(R_3 - R_{3c}), \tag{20}$$

where $E_4 = 0.55$, $R_{3c} = 10\ \mu$m, and the empirical multiplier γ, which varies from 0.01 and 0.1, was introduced to make the autoconversion rate smaller. The Heaviside function $H(R_3 - R_{3c})$ is again introduced to consider the threshold process such that the autoconversion rate is negligibly small when $R_3 < R_{3c}$.

In their GCM study, Boucher et al. (1995) assumed a fixed linear relation between R_4 and the mean volume radius, $R_4 = 1.1R_3$, and obtained an autoconversion parameterization given by

$$P_{Boucher} = \alpha_B N^{-1/3} L^{7/3} H(R_4 - R_{4c}), \tag{21a}$$

$$\alpha_B = \pi\kappa_1 \left(\frac{3}{4\pi\rho_w}\right)^{4/3} (1.1)^4 E_4 \gamma, \tag{21b}$$

where the Heaviside function $H(R_4 - R_{4c})$ is introduced to consider the threshold process such that the autoconversion rate is negligibly small when $R_4 < R_{4c}$. Note that the threshold process is determined by the mean radius of the fourth moment rather than the mean volume radius. Also noted is that, unlike Baker (1993), they found that a value of $\gamma = 1$ generated more reasonable results. A value of $E_4 = 0.55$ was used in this study. They also studied the sensitivity to the value of the threshold radius R_{4c}.

3. New parameterizations

a. Generalized R_4 parameterization

Compared to the Manton–Cotton parameterization, one of the features shared by the Baker and the Boucher parameterizations is that the mean volume radius R_3 in the Manton–Cotton parameterization is replaced by the mean radius of the fourth moment R_4 in both the conversion and the threshold coefficients. Values of the two average collection efficiencies, E_{MC} and E_4, may differ to some degree. These differences arise because the Baker and Boucher parameterizations eliminate the assumption of fixed terminal velocity. Furthermore, as will become evident later, the linear relation between R_4 and R_3 assumed in the Baker and Boucher parameterizations is easier to physically justify than the assumption that $R_{MC} = R_3$ in the Manton–Cotton parameterization. However, the differences between the three parameterizations are minimal in practice because the α parameters (α_{MC}, α_{Baker}, and $\alpha_{Boucher}$) and the threshold radii are arbitrarily tuned in most modeling studies. For this reason, the three parameterizations will hereafter be lumped together and referred to as the traditional R_4 parameterizations to emphasize the important role of the fourth moment [see Eq. (19)].

The Baker and the Boucher parameterizations can be generalized into a common expression by assuming a general linear relation between the mean volume radius and the mean radius of the fourth moment such that (see the appendix for details)

$$R_4 = \beta_4 R_3, \tag{22}$$

where β_4 is a nondimensional parameter depending on the spectral shape of the cloud droplet size distribution. Application of this expression gives the generalized R_4 parameterization

$$P_4 = \alpha_4 N^{-1/3} L^{7/3} H(R_4 - R_{4c}), \tag{23a}$$

$$\alpha_4 = \pi\kappa_1 \left(\frac{3}{4\pi\rho_w}\right)^{4/3} E_4 \beta_4^4. \tag{23b}$$

The differences between the three traditional R_4 parameterizations become evident from the above equations. The Baker parameterization is a special case of the generalized R_4 parameterization with $\beta_4 = 1$. In practice, the Manton–Cotton parameterization can also be considered a special case with $\beta_4 = 1$. As will be shown below, β_4 is an increasing function of the relative dispersion of the cloud droplet size distribution defined as the ratio of the standard deviation to the mean radius of the cloud droplet size distribution. A value of $\beta_4 = 1$ is equivalent to assuming a monodisperse cloud droplet size distribution. The Boucher parameterization corresponds to a special case of $\beta_4 = 1.1$, suggesting the assumption of a larger, yet fixed relative dispersion for the cloud droplet size distribution. Therefore, the primary differences between the traditional R_4 parameterizations reflect their different choices for the relative dispersion of the cloud droplet size distribution. Obviously, the assumption of fixed relative dispersion, monodisperse or not, is troublesome.

The gamma distribution $n(R) = N_0R^\mu \exp(-\lambda R)$ (N_0, μ, and λ are distribution parameters) have been widely used to describe cloud droplet size distributions in studies of the autoconversion rate (Berry 1967; Beheng 1994; Khairoutdinov and Kogan 2000). We have also found that observed cloud droplet size distributions are indeed well represented by the gamma distribution (Liu and Daum 2000a,b). For the gamma droplet size distribution, β_4 is easily shown to be uniquely related to the relative dispersion ε of the cloud droplet size distribution by (see the appendix for details)

$$\beta_4 = \frac{(1 + 3\varepsilon^2)^{1/4}}{[(1 + 2\varepsilon^2)(1 + \varepsilon^2)]^{1/12}}. \qquad (24)$$

According to this equation, $\beta_4 = 1.1$ used in the Boucher parameterization corresponds approximately to an $\varepsilon = 0.5$.

b. New R_6 parameterization

Although the various R_4 parameterizations are significant improvements of the original Kessler parameterization, they still suffer from the implicit deficiency that the collection efficiency is treated as a constant independent of droplet radius. This assumption is obviously incorrect because it means collections between droplets of nearly the same size are just as frequent as those between droplets of very different sizes. Baker (1993) discussed this deficiency and introduced a multiplicative parameter γ (from 0.01 to 0.1) to adjust for this effect. The other implicit assumption is the Stokes terminal velocity, although the effect of the latter assumption is minimal because the Stokes law describes terminal velocities of small droplets reasonably well. Here we develop a new parameterization that removes these assumptions.

Long (1974) showed that for $R < 50$ μm the collection kernel can be well approximated by

$$K(R, r) = \kappa_2 R^6, \qquad (25)$$

where the coefficient $\kappa_2 \approx 1.9 \times 10^{11}$ is in cm^{-3} s^{-1}, R is in cm, and the collection kernel K is in cm^3 s^{-1}. See also Pruppacher and Klett (1997) for more discussions on the Long kernel. Substitution of (25) into (3) yields

$$P_6 = \kappa_2 L \int R^6 n(R)\, dR = \kappa_2 N R_6^6 L, \qquad (26)$$

where N is in cm^{-3}, R_6 is the mean radius of the sixth moment in cm (see the appendix for the definition of R_6), L is in g cm^{-3}, and P_6 is g cm^{-3} s^{-1}. Similar to the generalized R_4 parameterization, we assume a general linear relation between the mean volume radius and the mean radius of the sixth moment, $R_6 = \beta_6 R_3$. This relation leads to the following expressions:

$$P_6 = \eta N^{-1} L^3 H(R_6 - R_{6c}), \qquad (27a)$$

$$\eta = \left(\frac{3}{4\pi\rho_w}\right)^2 \kappa_2 \beta_6^6, \qquad (27b)$$

where the Heaviside function $H(R_6 - R_{6c})$ is introduced to consider the threshold process such that the autoconversion rate is negligibly small when $R_6 < R_{6c}$. Note the change of the threshold radius from R_{4c} in the R_4 parameterizations to R_{6c} in the new R_6 parameterization.

For the purpose of comparison, the new R_6 parameterization is rewritten in the forms of the original Kessler and the R_4 parameterizations such that

$$P_6 = c_6 L H(R_6 - R_{6c}) = \alpha_6 N^{-1/3} L^{7/3} H(R_6 - R_{6c}), \qquad (28a)$$

$$c_6 = \alpha_6 N^{-1/3} L^{4/3}, \qquad (28b)$$

$$\alpha_6 = \left(\frac{3}{4\pi\rho_w}\right)^2 \kappa_2 \beta_6^6 \left(\frac{L}{N}\right)^{2/3}. \qquad (28c)$$

Again, under the assumption that the cloud droplet size distribution is described by the gamma distribution, the relationship between β_6 and the relative dispersion (ε) is easily shown to be

$$\beta_6 = \left[\frac{(1 + 3\varepsilon^2)(1 + 4\varepsilon^2)(1 + 5\varepsilon^2)}{(1 + \varepsilon^2)(1 + 2\varepsilon^2)}\right]^{1/6}. \qquad (29)$$

4. Comparisons of the Kessler-type parameterizations

To facilitate comparison, all the Kessler-type parameterizations discussed above are summarized in Table 1 in the forms of the Kessler parameterization and the R_4 parameterizations. Also given in the table are the major approximations and assumptions associated with these parameterizations as revealed by the common derivation. It is clear that the linear dependence of the autoconversion rate on the liquid water content in the original Kessler parameterization arises from the incorrect assumption of a fixed collection kernel. The R_4 parameterizations improve the original Kessler parameterization by relaxing the assumption of a fixed kernel to that of fixed collection efficiency. All of the R_4 parameterizations exhibit the same dependence on cloud liquid water content ($L^{7/3}$) and droplet concentration ($N^{-1/3}$); their differences lie in the characterization of the effect of the relative dispersion. The new R_6 parameterization exhibits an even stronger dependence on both the liquid water content (L^3) and the droplet concentration (N^{-1}) than the R_4 parameterizations. This improvement comes

TABLE 1. Summary of the Kessler-type autoconversion parameterizations, $P = cLH(y - y_c) = \alpha N^{-1/3}L^{7/3}H(y - y_c)$.

Parameterizations	Assumptions	Conversion coefficient c	y	y_c
Kessler	Fixed collection kernel	$c_K = a_K\left(1 - \frac{L_c}{L}\right)$	L	L_c
Manton–Cotton	Fixed collection efficiency, monodisperse spectrum, and fixed Stokes terminal velocity	$c_{MC} = \alpha_{MC}N^{-1/3}L^{4/3}$ $\alpha_{MC} = \pi\kappa_1\left(\frac{3}{4\pi\rho_w}\right)^{4/3}E_{MC}$	R_3	R_{3c}
Baker	Fixed collection efficiency, Stokes terminal velocity, and fixed, broader spectrum	$c_{Baker} = \alpha_{Baker}N^{-1/3}L^{4/3}$ $\alpha_{Baker} = \pi\kappa_1\left(\frac{3}{4\pi\rho_w}\right)^{4/3}E_4$	R_3	R_{3c}
Boucher	Fixed collection efficiency, Stokes terminal velocity, and fixed, broader spectrum	$c_{Boucher} = \alpha_{Boucher}N^{-1/3}L^{4/3}$ $\alpha_{Boucher} = \pi\kappa_1\left(\frac{3}{4\pi\rho_w}\right)^{4/3}E_4 \quad (1.1)$	R_4	R_{4c}
Generalized R_4	Fixed collection efficiency and Stokes terminal velocity	$c_4 = \alpha_4N^{-1/3}L^{4/3}$ $\alpha_4 = \pi\kappa_1\left(\frac{3}{4\pi\rho_w}\right)^{4/3}E_4\beta_4^4$	R_4	R_{4c}
New R_6	None of above	$c_6 = \alpha_6N^{-1/3}L^{4/3}$ $\alpha_6 = \kappa_1\left(\frac{3}{4\pi\rho_w}\right)^2\beta_6^6\left(\frac{L}{N}\right)^{2/3}$	R_6	R_{6c}

from the elimination of the incorrect assumption of fixed collection efficiency inherent in the R_4 parameterizations.

Examination of our new R_6 parameterization provides an explanation for a number of long-standing issues associated with the original Kessler parameterization as well as the various R_4 parameterizations. For example, such a wide range of values have been assigned to the coefficient a_K in studies using the original Kessler parameterization that, in practice, it has been often considered to be arbitrarily tunable (e.g., Kessler 1969; Liu and Orville 1969; Ghosh et al. 2000). It is evident from the new R_6 parameterization that the wide range of values assumed for a_K may stem from the variabilities in the liquid water content, droplet concentration, and relative dispersion that are not properly accounted for in the original Kessler parameterization. Similar to the arbitrary tunability of the coefficient a_K in the original Kessler parameterization, a wide range of values has been also assigned to the α (or γ) coefficient in modeling studies using the traditional R_4 parameterizations (Baker 1993; Boucher et al. 1995). For example, the range of γ from 0.01 to 1, as suggested by Baker (1993) and Boucher et al. (1995), alone leads to a difference of three orders of magnitude in α. The new R_6 parameterization again shows that the multitude of values that have been assigned to α may be due to the combined variabilities in liquid water content, droplet concentration, and relative dispersion.

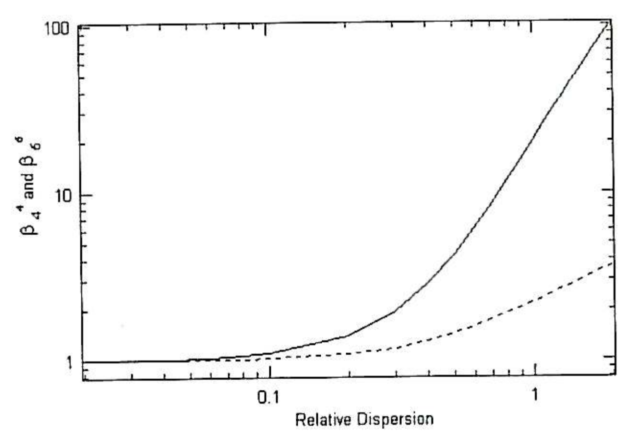

FIG. 1. Dependence of β_6^6 (solid line) and β_4^4 (dashed line) on the relative dispersion.

Both the generalized R_4 parameterization and the new R_6 parameterization explicitly account for the effect of the spectral shape through the dependency of β_4 and β_6 on the relative dispersion of the cloud droplet size distribution [(24) for β_4 and (29) for β_6]. This is a desirable feature because the effect of spectral shape on the autoconversion rate is well known, yet poorly quantified (Berry 1967, 1968; Berry and Reinhardt 1974a,b; Orville and Kopp 1977). However, the dependency of the autoconversion rate on the relative dispersion is quantitatively different for the generalized R_4 and the new R_6 parameterizations. Figure 1 shows β_4^4 and β_6^6 as a function of the relative dispersion within a range of values observed in ambient clouds (Liu and Daum 2000a,b, 2002). As shown in Fig. 1, both β_4^4 and β_6^6 increase with increasing relative dispersion, suggesting that the autoconversion rate is larger for a broader droplet size distribution when liquid water content and droplet concentration remain the same. It also suggests that the parameterizations assuming a monodisperse droplet size distribution underestimate the corresponding autoconversion rates and the degree of underestimation increases with the increasing relative dispersion because β_4^4 and β_6^6, respectively, represent the ratios of the autoconversion rates given by the generalized R_4 and the new R_6 parameterizations to their counterparts assuming a monodisperse droplet size distribution. Compared to the new R_6 parameterization, the generalized R_4 parameterization underestimates the effect of relative dispersion on the autoconversion rate as well, and the underestimation increases with the increasing relative dispersion. This deficiency of the generalized R_4 parameterization results from the incorrect assumption of fixed collection efficiency inherent in the generalized R_4 parameterization.

Relative dispersion also impacts the autoconversion rate by affecting the threshold radius. The threshold radius is determined by the mean volume radius R_3, by the mean radius of the fourth moment R_4, and by the mean radius of the sixth moment R_6 in the Manton–Cotton and the Baker parameterizations, in the Boucher and the generalized R_4 parameterizations, and in our new R_6 parameterization, respectively. It is noteworthy that R_6 is more closely related to the predominant radius that is shown to determine the collection process by Berry (1967, 1968) using a detailed microphysical model. Since the threshold radii for the generalized R_4 parameterization and the new R_6 parameterization are, respectively, given by $R_{4c} = \beta_4 R_{3c}$ and $R_{6c} = \beta_6 R_{3c}$, both parameterizations tend to have threshold radii larger than R_{3c} used in the Manton–Cotton and the Baker parameterizations. Simple calculations using (24) for β_4 and (29) for β_6 show that the differences between R_{3c}, R_{4c}, and R_{6c} increase with increasing relative dispersion. The underestimation by R_{3c} can reach up to a factor of 1.4 for the generalized R_4 and 2.0 for the new R_6 parameterizations, depending on the relative dispersion. According to few limited studies (Boucher et al. 1995), differences of such magnitude in threshold radii caused by the dispersion effect are large enough to significantly affect the results of climate simulations.

It is evident from the above discussion that the orginal Kessler parameterization and the various R_4 parameterizations have incorrect dependence of the autoconversion rate on liquid water content, droplet concentration, and relative dispersion due to using incorrect assumptions for the collection kernel. This fact should be emphasized because of their widespread use in cloud-related modeling ranging from LES models (Ghosh et al. 2000) to climate models (Boucher et al. 1995).

5. Comparison with model-based parameterizations

Over the last few decades, several empirical expressions have been developed to improve the parameterization of the autoconversion rate by curve-fitting the autoconversion rate obtained by numerically solving the stochastic collection equation under a variety of initial conditions (Berry 1968; Beheng 1994; Khairoutdinov and Kogan 2000). Similar to our simple R_6 parameterization, these model-based parameterizations all suggest stronger dependence of the autoconversion rate on liquid water content and droplet concentration than either the original Kessler parameterization or the various R_4 parameterizations.

Figure 2 shows a comparison of the new R_6 parameterization to the model-based parameterizations in more detail. A wide range of liquid water contents (from 0.01 to 5 g m^{-3}) and droplet concentrations (from 10 to 2500 cm^{-3}) are used in the calculation of the autoconversion rates presented in this figure, which covers virtually all the values likely observed in ambient clouds. Two points are obvious from Fig. 2. First, there are still significant discrepancies among the different model-based parameterizations. Second, the new R_6 parameterization well represents the average behavior of the different model-based parameterizations, lending additional support to the new R_6 parameterization. In addition, the analytically derived R_6 parameterization provides clearer physical insight than the model-based parameterizations in which physics is often blurred in the subtleties of the numerical model including initial conditions.

6. Conclusions

The Kessler-type autoconversion parameterizations that have been widely used in cloud-related modeling studies are theoretically derived and analyzed by applying the generalized mean value theorem for integrals to the general collection equation. The approximations implicitly assumed in these parameterizations, their logical connections, and the improvements are revealed by the derivations. It is shown that the original Kessler parameterization implicitly assumes a fixed collection kernel independent of droplet sizes. The Manton–Cotton parameterization improves the original Kessler parameterization by relaxing the assumptions of a fixed collection kernel to a fixed collection efficiency and fixed terminal velocity. The Baker and the Boucher parameterizations physically improve the Manton–Cotton parameterization by eliminating the assumption of a fixed terminal velocity. It is also demonstrated that the Manton–Cotton parameterization, the Baker parameterization, and the Boucher parameterization can be generalized to the so-called generalized R_4 parameterization, each being a special case of the generalized R_4 parameterization with a different, yet fixed relative dispersion.

A new R_6 parameterization is analytically derived by further eliminating the assumptions of fixed collection efficiency and the Stokes expression for the terminal velocity inherent in the various R_4 parameterizations by use of the Long kernel. The new R_6 parameterization

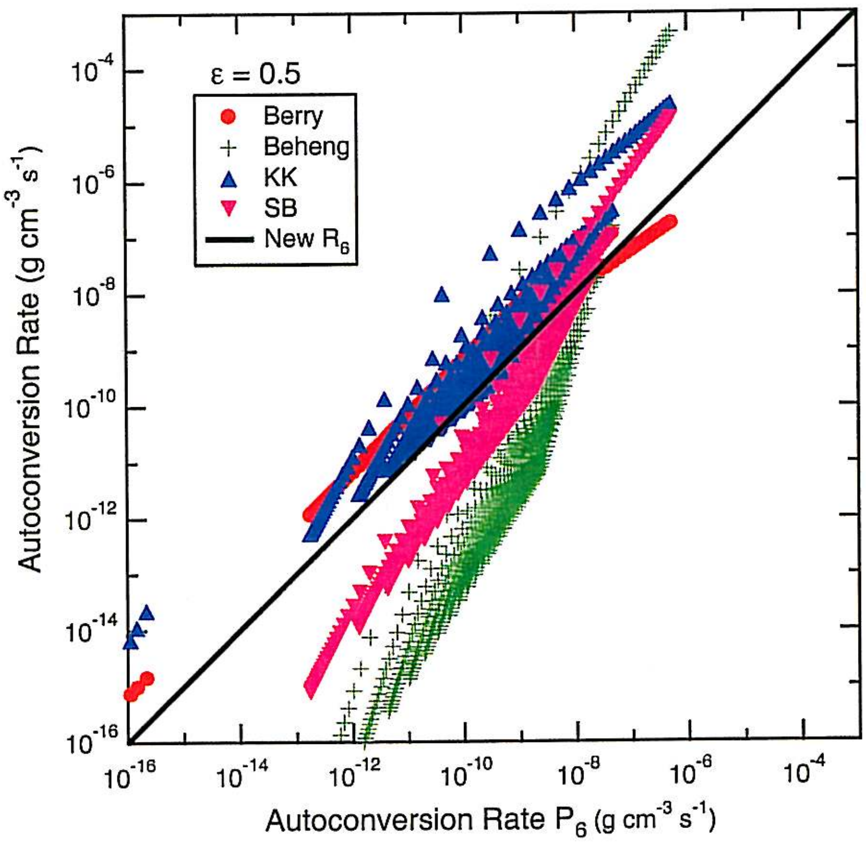

FIG. 2. Comparison of the new analytical R_6 parameterization (solid line) with previous model-based parameterizations, where KK represents the empirical parameterization given by Khairoutdinov and Kogan (2000) and SB represents a parameterization derived by Seifert and Beheng (2001) by analyzing the stochastic collection equation. Note that a relative dispersion of 0.50 as used in Boucher et al. (1995) is used in calculation of the autoconversion rate for parameterizations that explicitly account for the relative dispersion. Relative dispersion is not considered in the Khairoutdinov and Kogan parameterization. The threshold process embodied in the Heaviside function is not considered for the R_6 parameterization to maintain consistency with the parameterizations that do not consider the threshold process.

suggests stronger dependence of the autoconversion rate on liquid water content, droplet concentration, and relative dispersion and better represents the physics of the autoconversion process compared to previous Kessler-type parameterizations. This should be emphasized because the original Kessler parameterization and the various R_4 parameterizations are still being widely used despite their deficiencies clearly revealed in this work. Furthermore, examination of the new R_6 parameterization shows that the wide range of values chosen for both the a_K coefficient in the original Kessler parameterization and the α coefficient in the R_4 parameterizations may be mainly due to the combined variabilities in cloud liquid water content, droplet concentration, and relative dispersion in ambient clouds. The practice of arbitrarily tuning coefficients (a_K in the original Kessler parameterization and α in the R_4 parameterizations) to match some constraints in modeling studies is therefore misleading; critical information is lost in the tuning process. Further comparisons of the new R_6 parameterization with the existing model-based parameterizations lend additional support to the new R_6 parameterization.

In comparison with the commonly used Kessler-type parameterizations (the original Kessler parameterization and the traditional R_4 parameterizations), the new R_6 parameterization has an additional advantage because it includes relative dispersion as a dependent variable and can be used to study the effect of the relative dispersion on the autoconversion rate. Our preliminary analysis indicates that the effect of the relative dispersion needs to be accounted for in the parameterization of the autoconversion process. The importance of the relative dispersion is further reinforced by its substantial effect on cloud radiative properties (Liu and Daum 2000a,b; Wood 2000). It has been recently shown that the relative dispersion is enhanced by anthropogenic aerosols and that this enhanced dispersion significantly affects the evaluation of the Twomey effect (Liu and Daum 2002; Peng and Lohmann 2003; Rotstayn and Liu 2003). It is expected that the enhanced relative dispersion will further affect the evaluation of the so-called second indirect aerosol effect by affecting the autoconversion rate. The dispersion-dependent R_6 parameterization can be used to quantify this effect. It is noteworthy that some model-based parameterizations (e.g., the Berry and the Beheng parameterizations) also consider the relative dispersion as a dependent variable.

Several points are noteworthy in passing. First, cloud turbulence is also known to affect the dispersion of the cloud droplet size distribution and the collection process (Telford 1996; Liu et al. 2002; Shaw 2003). Thus, further development of the autoconversion parameterization must consider cloud turbulence as well. Second, some studies (Cotton 1972; Seifert and Beheng 2001) suggest that the transient (or aging) behavior of the autoconversion process could be important; more research on this issue is needed. Third, similar parameterizations for the autoconversion rate have been used in models of various scales ranging from LES models to global climate models, and this brings up the issue of the scale dependence of such parameterizations (Pincus and Klein 2000; Rotstayn 2000; Wood et al. 2002; Zhang et al. 2002). Approaches other than the simple Kessler-type parameterizations may be necessary for treating these complex problems. A potential candidate was proposed by Feingold et al. (1998), which represents cloud and drizzle size distributions using two analytical functions (e.g., lognormal).

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APPENDIX

Various Mean Radii and Expressions for β_4 and β_6

The pth moment of the cloud droplet size distribution $n(R)$ is defined as

$$M_p = \frac{\int R^p n(R)\, dR}{N}, \qquad \text{(A1)}$$

where N is the total droplet number concentration, and $p = 1, 2, 3, \ldots$. The mean radius of the pth moment is defined as

$$R_p = M_p^{1/p}. \qquad \text{(A2)}$$

For the commonly used gamma droplet size distributions $n(R) = N_0 R^\mu \exp(-\lambda R)$, R_p is given by

$$R_p = \left(\frac{1}{\lambda}\right)\left[\frac{\Gamma(p + \mu + 1)}{\Gamma(1 + \mu)}\right]^{1/p}, \qquad \text{(A3)}$$

where $\Gamma(\cdot)$ is the gamma function. Therefore, we have

$$\beta_p = \frac{R_p}{R^3} = \left[\frac{\Gamma(p + \mu + 1)}{\Gamma(1;+\text{th}\,\mu)}\right]^{1/p}\left[\frac{\Gamma(3 + \mu + 1)}{\Gamma(1 + \mu)}\right]^{-1/3}, \qquad \text{(A4)}$$

$$\beta_4 = \frac{(4 + \mu)^{1/4}}{[(3 + \mu)(2 + \mu)(1 + \mu)]^{1/12}}, \quad \text{and} \qquad \text{(A5)}$$

$$\beta_6 = \frac{[(6 + \mu)(5 + \mu)(4 + \mu)]^{1/6}}{[(3 + \mu)(2 + \mu)(1 + \mu)]^{1/6}}. \qquad \text{(A6)}$$

For the gamma distribution, the spectral shape parameter μ is related to the relative dispersion by

$$\mu = \varepsilon^{-2} - 1. \qquad \text{(A7)}$$

Substitution of (A7) into (A5) and (A6), respectively, yields

$$\beta_4 = \frac{(1 + 3\varepsilon^2)^{1/4}}{[(1 + 2\varepsilon^2)(1 + \varepsilon^2)]^{1/12}} \quad \text{and} \qquad \text{(A8)}$$

$$\beta_6 = \left[\frac{(1 + 3\varepsilon^2)(1 + 4\varepsilon^2)(1 + 5\varepsilon^2)}{(1 + \varepsilon^2)(1 + 2\varepsilon^2)}\right]^{1/6}. \qquad \text{(A9)}$$

It is noteworthy that, besides the gamma distribution, a lognormal distribution and Weibull distribution are often used to describe droplet size distributions as well. For these distributions, R_p is linearly related to R_3, but with different dependences of β_p on the relative dispersion. The gamma distribution is chosen here because it describes observed droplet size distributions better than the lognormal distribution and gives simple analytical expressions for β_4 and β_6 (Liu and Daum 2000b). Detailed comparisons will be addressed elsewhere.

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Reply

We welcome and appreciate the comment by Wood and Blossey (2005, hereafter WB) on our paper (Liu and Daum 2004, hereafter LH). In response, we would like to make the following points.

First, as clearly stated in LH, Kessler-type parameterizations take the form of a product of two different functions:

$$P = P_0 H\left(r_d - r_c\right), \tag{1}$$

where P_0 represents the autoconversion rate after the onset of the autoconversion process (autoconversion function hereafter), and the Heaviside step function H represents the threshold behavior such that the autoconversion rate is zero when the driving radius r_d is less than the critical radius r_c. Differences between the various Kessler-type parameterizations arise from how P_0 and r_d are specified. In the Liu-Daum parameterization, both P_0 and r_d are determined by the mean radius of the sixth moment of the cloud droplet size distribution instead of the mean radius of the third or fourth moment of previous parameterizations. It is noteworthy that WB essentially compares the autoconversion rate calculated from a detailed collection model to the autoconversion function P_0, not the Liu-Daum parameterization, which is the product of P_0 and the Heaviside step function introduced to represent the threshold behavior. Furthermore, as demonstrated in WB's Fig. 2b, WB tends to focus more on the threshold behavior. This figure shows the ratio of P_0 to P as a function of the volume-mean radius, and reveals that the real autoconversion rate falls sharply after the driving radius r_d is less than some threshold value between 10 and 15 μm.

Second, there are two different approaches that have been used to mathematically define the autoconversion rate. According to Kessler's original ideas, autoconversion starts once some threshold is crossed, and the autoconversion rate represents the growth rate by the collection process integrated over drops from the critical radius to sizes that are large enough to fall as small raindrops. Existing Kessler-type parameterizations, including the Liu-Daum parameterization derived in LH, basically follow this definition, and assume an abrupt threshold behavior described by the Heaviside step function. The other approach, pioneered by Beheng and his group (e.g., Beheng 1994), is used in WB. The Beheng approach separates self-collection of cloud droplets (collected cloud droplets remain as cloud droplets) from the autoconversion process, and seems reasonable at first glance. However, as correctly pointed out in WB, the result obtained using this approach is highly sensitive to the separation radius r_0. Separation radius is introduced to distinguish cloud droplets from raindrops, but there appears to be no fundamental basis for choosing a value for it, and values from 20 μm (e.g., WB) to 50 μm (e.g., Beheng 1994) to 100 μm (Simpson and Wiggert 11969) have been used. Note that WB appears to confuse the separation radius with the critical radius as defined in Liu et al. (2004) and McGraw and Liu (2004).

Finally, the results presented in WB indeed raise questions as to the representation of the threshold behavior and the effect of truncating the cloud droplet size distribution on the autoconversion rate. It is clear from WB's Fig. 2b that the "all-or-nothing" representation of the threshold behavior by the Heaviside step function used in Kessler-type parameterizations, including the Liu-Daum parameterization, does not accurately describe the threshold behavior; the change of the autoconversion rate near the threshold is smooth, not discontinuous as characterized by the Heaviside step function. Therefore, to further improve the autoconversion parameterization requires going beyond the commonly used Kessler-type parameterizations. Another related issue is the choice between the two different definitions of the autoconversion rate, which should be consistent with the other processes (e.g., accretion) that need to be parameterized in atmospheric models.

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