BNL-75680-2006-JA-R1

GEOPHYSICAL RESEARCH LETTERS, VOL. 33, L11804, doi:10.1029/2005GL025500, 2006

Generalized threshold function accounting for effect of relative dispersion on threshold behavior of autoconversion process

Yangang Liu,[1] Peter H. Daum,[1] Robert McGraw,[1] and Mark Miller[1]

Received 15 December 2005; revised 3 March 2006; accepted 3 April 2006; published 2 June 2006.

[1] The recently derived theoretical threshold function associated with the autoconversion process is generalized to account for the effect of the relative dispersion of the cloud droplet size distribution. This generalized threshold function theoretically demonstrates that the relative dispersion, which has been largely neglected to date, essentially controls the cloud-to-rain transition if the liquid water content and the droplet concentration are fixed. Comparison of the generalized threshold function to existing ad hoc threshold functions further reveals that the essential role of the spectral shape of the cloud droplet size distribution in rain initiation has been unknowingly buried in the arbitrary use of ad hoc threshold functions in atmospheric models such as global climate models, and that commonly used ad hoc threshold functions are unable to fully describe the threshold behavior of the autoconversion process that likely occurs in ambient clouds. **Citation:** Liu, Y., P. H. Daum, R. McGraw, and M. Miller (2006), Generalized threshold function accounting for effect of relative dispersion on threshold behavior of autoconversion process, *Geophys. Res. Lett.*, *33*, L11804, doi:10.1029/2005GL025500.

1. Introduction

[2] The autoconversion process whereby cloud droplets grow into embryonic raindrops is a key microphysical process that needs to be parameterized in atmospheric models such as cloud resolving models and global climate models [*Kessler*, 1969; *Manton and Cotton*, 1977; *Liou and Ou*, 1989; *Baker*, 1993; *Liu and Daum*, 2004]. Accurate parameterization of the autoconversion process is especially important for estimating the second indirect aerosol effect [*Boucher et al.*, 1995; *Lohmann and Feichter*, 1997; *Rotstayn*, 2000; *Rotstayn and Liu*, 2005].

[3] All the autoconversion parameterizations that have been developed so far can be generically written as

$$P = P_0 T, \tag{1}$$

where P is the autoconversion rate; P_0 is the rate function describing the conversion rate after the onset of the autoconversion process, and T is the threshold function describing the threshold behavior of the autoconversion process. The rate function P_0 has been the primary focus of previous studies, and great progress has been made over the last few decades [*Kessler*, 1969; *Manton and Cotton*, 1977; *Liou and Ou*, 1989; *Baker*, 1993; *Liu and Daum*, 2004; *Chen and Liu*, 2004; *Wood*, 2005]. The threshold function, however, has received little attention, and the commonly used threshold functions are ad hoc in nature [*Kessler*, 1969; *Sundqvist*, 1978; *Del Genio et al.*, 1996; *Liu et al.*, 2006a].

[4] We have recently derived a theoretical threshold function by truncating the collection equation at the critical radius (LDM threshold function) [*Liu et al.*, 2005]. Although the LDM threshold function provides a firm physical basis for the threshold behavior of the autoconversion process, it only considers the liquid water content (L) and the droplet concentration (N) as independent variables, and implicitly assumes a constant relative dispersion (ε, defined as the ratio of standard deviation to the mean radius of the cloud droplet size distribution). The assumption of a constant ε is a drawback of the LDM threshold function, because the spectral shape of the droplet size distribution is expected to vary in ambient clouds and to have a significant effect on rain initiation [*Hudson and Yum*, 1997]. Furthermore, both observational and theoretical evidence indicates that increasing aerosols concurrently increase N and ε, and the enhanced ε leads to a warming dispersion effect on climate [*Liu and Daum*, 2002; *Rotstayn and Liu*, 2003; Peng and Lohmann, 2003; *Liu et al.*, 2006b]. Without explicit specification of ε, the LDM threshold function is handicapped in applications such as investigating rain initiation and the second indirect aerosol effect [*Rotstayn and Liu*, 2005].

[5] The primary objective of this work is to generalize the LDM threshold function to account explicitly for ε in addition to L and N, and to use this new generalized threshold function to examine commonly used ad hoc threshold functions.

2. LDM Threshold Function and Its Generalization

[6] According to *Liu et al.* [2005], the threshold function can be generally described by

$$T = \frac{P}{P_0} = \left[\frac{\int_{r_c}^{\infty} r^6 n(r)dr}{\int_0^{\infty} r^6 n(r)dr}\right]\left[\frac{\int_{r_c}^{\infty} r^3 n(r)dr}{\int_0^{\infty} r^3 n(r)dr}\right], \tag{2}$$

where r is the droplet radius, n(r) is the cloud droplet size distributions, and r_c is the critical radius. Under the assumption that the cloud droplet size distribution is described by

$$n(r) = \frac{3N}{r_3^3} r^2 \exp\left[-\left(\frac{r}{r_3}\right)^3\right], \tag{3}$$

[1]Brookhaven National Laboratory, Upton, New York, USA.

0094-8276/06/2005GL025500$05.00

the LDM threshold function was derived to be

$$\mathrm{T_{LDM}} = \frac{1}{2}\left(x_c^2 + 2x_c + 2\right)(1 + x_c)\mathrm{e}^{-2x_c}, \tag{4a}$$

where r_3 is the mean-volume radius, and the critical-to-mean mass ratio x_c is a function of L and N given by [*McGraw and Liu*, 2003, 2004; *Liu et al.*, 2004, 2005]

$$x_c = 9.7 \times 10^{-17} N^{3/2} L^{-2}. \tag{4b}$$

The LDM threshold function holds only for a special Weibull size distribution that is described by equation (3) with ε = 0.36, and hence does not consider ε as an independent variable.

[7] To incorporate ε, we replace equation (3) with the general Weibull size distribution

$$n(r) = \frac{q\Gamma^{q/3}\left(\frac{3+q}{q}\right)N}{r_3^q} r^{q-1} \exp\left[-\Gamma^{q/3}\left(\frac{3+q}{q}\right)\left(\frac{r}{r_3}\right)^q\right], \tag{5}$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the Gamma function. The parameter q describes the spectral shape of the cloud droplet size distribution, and is related to ε by [*Liu and Daum*, 2000]

$$\varepsilon = \left[\frac{2q\Gamma(2/q)}{\Gamma^2(1/q)} - 1\right]^{1/2}. \tag{6}$$

Theoretical and observational justification for using the general Weibull droplet distribution are given by *Liu et al.* [1995], *Liu and Hallett* [1997], and *Liu and Daum* [2000]. Substitution of equation (5) into equation (2) and subsequent integration yields the generalized threshold function

$$T_q = \frac{\Gamma\left(\frac{6+q}{q}, x_{cq}\right)\Gamma\left(\frac{3+q}{q}, x_{cq}\right)}{\Gamma\left(\frac{6+q}{q}\right)\Gamma\left(\frac{3+q}{q}\right)} = \gamma\left(\frac{6+q}{q}, x_{cq}\right)\gamma\left(\frac{3+q}{q}, x_{cq}\right), \tag{7a}$$

$$x_{cq} = \left(\frac{r_c}{r_0}\right)^q = \Gamma^{q/3}\left(\frac{3+q}{q}\right)x_c^{q/3}. \tag{7b}$$

where $\Gamma(t, z) = \int_z^\infty x^{t-1}e^{-x}dx$ is the incomplete Gamma function, and $\gamma()$ denotes the incomplete Gamma function normalized by the corresponding complete Gamma function [*Press et al.*, 1992]. Equations (7a) and (7b) show that T_q is determined by x_c and q, and reduces to the LDM threshold function when q = 3.

[8] Although T_q as given by a combination of equations (6) and (7) quantifies the dependence of the threshold behavior on ε, the dependence has to be determined by repeating the procedure of calculating T_q and ε for different values of q. This is not an ideal feature for application in atmospheric models, which prefer simple relationships.

[9] *Liu et al.* [2002] showed that equation (6) is well approximated by

$$q \approx \varepsilon^{-1}. \tag{8}$$

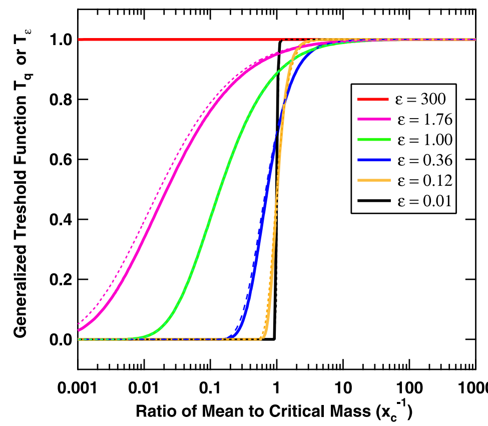

Figure 1. The generalized threshold function T_q as a function of the mean-to-critical mass ratio (x_c^{-1}). The solid and dotted lines represent results calculated from T_q and its approximation T_ε, respectively. Note that the approximation is so accurate that most curves of T_ε overlaps with those of T_q, except those with the relative dispersion ε = 1.76.

It is noteworthy that equation (8) actually gives the exact results for ε = 0, 1, and ∞, which corresponds to q = ∞, 1, and 0, respectively. Substitution of equation (8) into equation (7) yields

$$T_\varepsilon = \gamma\left(\frac{6+\varepsilon^{-1}}{\varepsilon^{-1}}, \Gamma^{1/(3\varepsilon)}\left(\frac{3+\varepsilon^{-1}}{\varepsilon^{-1}}\right)x_c^{1/(3\varepsilon)}\right) \cdot \gamma\left(\frac{3+\varepsilon^{-1}}{\varepsilon^{-1}}, \Gamma^{1/(3\varepsilon)}\left(\frac{3+\varepsilon^{-1}}{\varepsilon^{-1}}\right)x_c^{1/(3\varepsilon)}\right). \tag{9}$$

[10] The above theoretical analysis shows that the threshold function is determined by two dimensionless quantities: x_c and ε. Figure 1 shows T_q (solid lines) as a function of the mean-to-critical mass ratio, i.e., the reciprocal of x_c, at different values of ε. Also shown is T_ε (dotted lines) to compare its performance with T_q. Two points are evident from this figure. First, the dependence of the threshold behavior on the mean-to-critical mass ratio (x_c^{-1}) gradually changes from a constant of T = 1 to a discontinuous δ-function as ε decreases from ∞ to 0. When $x_c^{-1} < 1$, a larger ε leads to a larger value of the threshold function, and the same amount of water converted from cloudwater to rainwater requires a smaller x_c (higher liquid water content and/or lower droplet concentration) for a smaller ε. This behavior is largely due to the enhanced collection process resulting from a broader droplet size distribution. These results highlight the importance of ε in rain initiation, and are consistent with the microphysical theory that droplet collision requires relative velocities resulting from droplets of different sizes [*Pruppacher and Klett*, 1997]. Second, T_ε is an excellent approximation of T_q for virtually all the combinations of ε and x_c and can be used as a substitute for T_q in practice (note that all the dashed T_ε lines but that for ε = 1.76 overlap with the solid T_q lines).

3. Application of T_q to Examining Ad Hoc Threshold Functions

[11] To obtain physical understanding of the commonly used ad hoc threshold functions, this section examines them

by comparing to the theoretically derived generalized threshold function. Traditional ad hoc threshold functions can be generally classified as Kessler-type, Berry-type or Sundqvist-type according to their mathematical form. Briefly, the Kessler-type threshold function is a Heaviside step function [*Manton and Cotton*, 1977; *Liou and Ou*, 1989; *Baker*, 1993; *Liu and Daum*, 2004]

$$T_K = H(r_m - r_c), \tag{10}$$

where r_m and r_c denotes the driving and critical radii, respectively. The other extreme is often associated with empirical autoconversion parameterizations obtained by fitting simulations from detailed microphysical models, which generally have no threshold functions, or implicitly assume a constant threshold function [*Berry*, 1968; *Beheng*, 1994; *Khairoutdinov and Kogan*, 2000], i.e.,

$$T_B = 1. \tag{11}$$

According to the discussion in Section 2, the Kessler-type and Berry-type threshold functions in fact only represent monodisperse and extremely broad cloud droplet size distributions, respectively. Furthermore, because the two extreme spectral shapes are unlikely to occur in ambient clouds [*Liu and Daum*, 2000], neither the Kessler-type nor the Berry-type threshold functions are generally applicable.

[12] The Sundqvist-type threshold function lies between the two extremes [*Sundqvist*, 1978; *Del Genio et al.*, 1996]. However, traditional Sundqvist-type parameterizations account only for L, and are therefore unsuitable for studies of the second indirect aerosol effect. *Liu et al.* [2006a] has recently proposed a generalized Sundqvist-type threshold function

$$T_S = 1 - \exp\left[-x_c^{-\mu}\right], \tag{12}$$

where $\mu \geq 0$ is an empirical exponent. The generalized Sundqvist-type threshold function not only exhibits a smooth threshold behavior, but also encompasses virtually all the ad hoc threshold functions: it reduces to the traditional Sundqvist-type threshold functions proposed by Sundqvist and Del Genio when $\mu = 2$ and 4, respectively, approximately becomes the Berry-type (T = 0.63 not 1) when $\mu = 0$, and approaches the Kessler-type when μ approaches $\propto$ (see *Liu et al.* [2006a] for detailed discussion). Evidently, T_s is an improvement over the traditional ad hoc threshold functions that only work for some special spectral shapes of the droplet size distribution. Nevertheless, there is no physical basis for T_s, and especially, the physical meaning of μ is elusive.

[13] Careful comparison of T_s to T_q further indicates that T_s approximately describes the overall threshold behavior of T_q, if μ is a decreasing function of ε [A trial-and-error analysis suggests that $\mu = \varepsilon^{-1}$ is not a bad approximation]. Unfortunately, Despite the improvement over the Kessler-type and Berry-type threshold functions, T_s can accurately describe the threshold behavior only for narrow droplet size distributions, and is not suitable for broad droplet size distributions in ambient clouds.

[14] Furthermore, because ε likely varies from 0.1 to 10 in ambient clouds [*Liu and Daum*, 2000; *Wood*, 2000; *McFarquhar and Heymsfield*, 2001], the above analyses suggest that all the commonly used ad hoc threshold functions cannot adequately describe the real threshold behavior of the autoconversion process that occurs in nature.

4. Concluding Remarks

[15] The theoretical threshold function associated with the autoconversion process presented by *Liu et al.* [2005] is generalized to account explicitly for the effect of the relative dispersion of the cloud droplet size distribution. The generalized threshold function theoretically shows that the relative dispersion, which has been largely neglected to date, essentially controls the initial transition from cloud-water to rainwater for fixed liquid water contents and the droplet concentrations. Comparison of the generalized threshold function with existing ad hoc threshold functions shows that the threshold behavior of the autoconversion process changes from the Berry-type to the Sundqvist-type to the Kessler-type as the relative dispersion decreases, and that the empirical parameter μ in the generalized Sundqvist-type threshold function is a decreasing function of the relative dispersion, providing physical explanations for these strikingly different ad hoc threshold functions. The comparison also suggests that commonly used ad hoc threshold functions only well describe the threshold behavior for some special spectral shapes of the cloud droplet size distribution, and cannot fully describe the threshold behavior that likely occurs in ambient clouds. These results indicate that the important role of ε in rain initiation has been unknowingly buried in the traditional practice of arbitrarily choosing ad hoc threshold functions.

[16] It is noted that although the importance of the spectral shape of the cloud droplet size distribution in rain initiation has been long recognized, the spectral shape effect has been poorly understood and quantified in atmospheric models. A recent study has demonstrated that modeling results are very sensitive to the treatment of the threshold function [*Rotstayn and Liu*, 2005]. It will be interesting to examine the effect of the relative dispersion on modeling results using this generalized threshold function. The explicit consideration of the relative dispersion in the generalized threshold function also allows for evaluation of the effect of the relative dispersion caused by anthropogenic aerosols on rain initiation, and the second indirect aerosol effect. Furthermore, the high sensitivity of the threshold behavior to the relative dispersion reinforces the need to account explicitly for the relative dispersion in the parameterization of the autoconversion process, which is still in its infancy [*Liu and Daum*, 2000; *Liu et al.*, 2006b].

[17] **Acknowledgments.** The authors thank Y. Peng at Max Planck Institute of Meteorology and the anonymous reviewers for their insightful comments. This research was supported by Atmospheric Radiation Measurements Program and Atmospheric Sciences Program of the U.S. Department of Energy. The United States Government retains, and the publisher, by accepting the article for publications, acknowledges, a world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for the United States Government purposes.

References

Baker, M. B. (1993), Variability in concentrations of CCN in the marine cloud-top boundary layer, *Tellus, Ser. B*, *45*, 458–472.

Beheng, K. D. (1994), A parameterization of warm cloud microphysical conversion processes, *Atmos. Res.*, *33*, 193–206.

Berry, E. X. (1968), Modification of the warm rain process, paper presented at 1st National Conference on Weather Modification, Am. Meteorol. Soc., Albany, N. Y.

Boucher, O., H. L. Treut, and M. B. Baker (1995), Precipitation and radiation modeling in a general circulation model: Introduction of cloud microphysical process, *J. Geophys. Res.*, *100*, 16,395–16,414.

Chen, J., and S. Liu (2004), Physically based two-moment bulkwater parameterizations for warm-cloud microphysics, *Q. J. R. Meteorol. Soc.*, *130*, 51–78.

Del Genio, A. D., M. Yao, W. Kovari, and K. K. Lo (1996), A prognostic cloud water parameterization for climate models, *J. Clim.*, *9*, 270–304.

Hudson, J. G., and S. S. Yum (1997), Droplet spectral broadening in marine stratus, *J. Atmos. Sci.*, *54*, 2642–2654.

Kessler, E. (1969), On the distribution and continuity of water substance in atmospheric circulation, *Meteorol. Monogr.*, *10*, 84 pp.

Khairoutdinov, M., and Y. Kogan (2000), A new cloud physics parameterization in a large-eddy simulation model of marine stratocumulus, *Mon. Weather Rev.*, *128*, 229–243.

Liou, K. N., and S. C. Ou (1989), The role of cloud microphysical processes in climate: An assessment from a one-dimensional perspective, *J. Geophys. Res.*, *94*, 8599–8607.

Liu, Y., and P. H. Daum (2000), Spectral dispersion of cloud droplet size distributions and the parameterization of cloud droplet effective radius, *Geophys. Res. Lett.*, *27*, 1903–1906.

Liu, Y., and P. H. Daum (2002), Indirect warming effect from dispersion forcing, *Nature*, *419*, 580–581.

Liu, Y., and P. H. Daum (2004), Parameterization of the autoconversion process. part I: Analytical formulation of the Kessler-type parameterizations, *J. Atmos. Sci.*, *61*, 1539–1548.

Liu, Y., and J. Hallett (1997), The "1/3" power-law between effective radius and liquid water content, *Q. J. R. Meteorol. Soc.*, *123*, 1789–1795.

Liu, Y., L. You, W. Yang, and F. Liu (1995), On the size distribution of cloud droplets, *Atmos. Res.*, *35*, 201–216.

Liu, Y., P. H. Daum, S. K. Chai, and F. Liu (2002), Cloud parameterizations, cloud physics, and their connections: An overview, *Recent Res. Dev. Geophys.*, *4*, 119–142.

Liu, Y., P. H. Daum, and R. McGraw (2004), An analytical expression for predicting the critical radius in the autoconversion parameterization, *Geophys. Res. Lett.*, *31*, L06121, doi:10.1029/2003GL019117.

Liu, Y., P. H. Daum, and R. McGraw (2005), Size truncation effect, threshold behavior, and a new type of autoconversion parameterization, *Geophys. Res. Lett.*, *32*, L11811, doi:10.1029/2005GL022636.

Liu, Y., P. H. Daum, and R. McGraw (2006a), Parameterization of the autoconversion process. Part II: Generalization of Sundqvist-type parameterizations, *J. Atmos. Sci.*, *63*, 1103–1109.

Liu, Y., P. H. Daum, and S. Yum (2006b), Analytical expression for the relative dispersion of the cloud droplet size distribution, *Geophys. Res. Lett.*, *33*, L02810, doi:10.1029/2005GL024052.

Lohmann, U., and J. Feichter (1997), Impact of sulfate aerosols on albedo and lifetime of clouds: A sensitivity study with the ECHAM 4 GCM, *J. Geophys. Res.*, *102*, 13,685–13,700.

Manton, M. J., and W. R. Cotton (1977), Formulation of approximate equations for modeling moist deep convection on the mesoscale, Atmos. Sci. Pap. 266, 62 pp., Dep. of Atmos. Sci., Colo. State Univ., Boulder.

McFarquhar, G. M., and A. J. Heymsfield (2001), Parameterizations of INDOEX microphysical measurements and calculations of cloud susceptibility: Applications for climate studies, *J. Geophys. Res.*, *106*, 28,675–28,698.

McGraw, R., and Y. Liu (2003), Kinetic potential and barrier crossing: A model for warm cloud drizzle formation, *Phys. Rev. Lett.*, *90*, doi:10.1103/PhysRevLett.90.018501.

McGraw, R., and Y. Liu (2004), Analytical formulation and parameterization of the kinetic potential theory for drizzle formation, *Phys. Rev. E*, *70*, doi:10.1103/PhysRevE.70.031606.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery (1992), *Numerical Recipes in FORTRAN*, 963 pp., Cambridge Univ. Press, New York.

Pruppacher, H. R., and J. D. Klett (1997), *Microphysics of Clouds and Precipitation*, 954 pp., Springer, New York.

Rotstayn, L. D. (2000), On the "tuning" of autoconversion parameterizations in climate models, *J. Geophys. Res.*, *105*, 15,495–15,508.

Rotstayn, L. D., and Y. Liu (2003), Sensitivity of the first indirect aerosol effect to an increase of cloud droplet spectral dispersion with droplet number concentration, *J. Clim.*, *16*, 3476–3481.

Rotstayn, L. D., and Y. Liu (2005), A smaller global estimate of the second indirect aerosol effect, *Geophys. Res. Lett.*, *32*, L05708, doi:10.1029/2004GL021922.

Sundqvist, H. (1978), A parameterization scheme for non-convective condensation including prediction of cloud water content, *Q. J. R. Meteorol. Soc.*, *104*, 677–690.

Wood, R. (2000), Parameterization of the effect of drizzle upon the droplet effective radius in stratocumulus clouds, *Q. J. R. Meteorol. Soc.*, *126*, 3309–3324.

Wood, R. (2005), Drizzle in stratiform boundary layer clouds. part II: Microphysical aspects, *J. Atmos. Sci.*, *62*, 3034–3050.

P. H. Daum, Y. Liu, R. McGraw, and M. Miller, Brookhaven National Laboratory, Bldg. 815E, Upton, NY 11973–5000, USA. (lyg@bnl.gov)