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BEHAVIOR OF AUTOCONVERSION PROCESS***

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Generalized Threshold Function Accounting for Effect of Relative Dispersion on Threshold Behavior of Autoconversion Process

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Abstract. The recently derived theoretical threshold function associated with the autoconversion process is generalized to account for the effect of the relative dispersion of the cloud droplet size distribution. This generalized threshold function theoretically demonstrates that the relative dispersion, which has been largely neglected to date, essentially controls the cloud-to-rain transition if the liquid water content and the droplet concentration are fixed. Comparison of the generalized threshold function to existing ad hoc threshold functions further reveals that the essential role of the spectral shape of the cloud droplet size distribution in rain initiation has been unknowingly buried in the arbitrary use of ad hoc threshold functions in atmospheric models such as global climate models, and that none of the commonly used ad hoc threshold functions accurately describe the threshold behavior of the autoconversion process that likely occurs in ambient clouds.

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1. Introduction

The autoconversion process whereby cloud droplets grow into embryonic raindrops is a key microphysical process that needs to be parameterized in atmospheric models such as cloud resolving models and global climate models [Kessler, 1969; Manton and Cotton, 1977; Liou and Ou, 1989; Baker, 1993; Liu and Daum, 2004]. Accurate parameterization of the autoconversion process is especially important for estimating the second indirect aerosol effect. [Boucher et al., 1995; Lohmann and Fleichter, 1997; Rotstayn, 2000; Rotstayn and Liu, 2005].

All the autoconversion parameterizations that have been developed so far can be generically written as

$$P = TP_0, \quad [1]$$

where P is the autoconversion rate; P_0 is the rate function describing the conversion rate after the onset of the autoconversion process, and T is the threshold function describing the threshold behavior of the autoconversion process. The rate function P_0 has been the primary focus of previous studies, and great progress has been made over the last few decades [Kessler, 1969; Manton and Cotton, 1977; Liou and Ou, 1989; Baker, 1993; Liu and Daum, 2004; Chen and Liu, 2004; Wood, 2005]. The threshold function, however, has received little attention, and the commonly used threshold functions are ad hoc in nature [Kessler, 1969; Sundqvist, 1978; Del Genio et al., 1996; Liu et al. 2006a].

We have recently derived a theoretical threshold function by truncating the collection equation at the critical radius (LDM threshold function hereafter, Liu et al., 2005). Although the LDM threshold function provides a firm physical basis for the

threshold behavior of the autoconversion process, it only considers liquid water content (L) and the droplet concentration (N) as independent variables, and assumes a constant relative dispersion (ε , defined as the ratio of standard deviation to the mean radius of the cloud droplet size distribution). The assumption of a constant ε is a drawback of the LDM threshold function, because the spectral shape of the droplet size distribution is expected to vary in ambient clouds and to have a significant effect on rain initiation [Hudson and Yum, 1997]. Furthermore, both observational and theoretical evidence indicates that increasing aerosols concurrently increase N and ε , and the enhanced ε leads to a warming dispersion effect on climate [Liu and Daum, 2002; Rostyan and Liu, 2003; Peng and Lohmann, 2003; Liu et al., 2006b]. Without explicit specification of ε , the LDM threshold function is handicapped in applications such as investigating rain initiation and the second indirect aerosol effect [Rotstyan and Liu, 2005].

The primary objective of this work is to generalize the LDM threshold function to account explicitly for ε in addition to L and N, and to use this new generalized threshold function to examine commonly used ad hoc threshold functions.

2. LDM Threshold Function and Its Generalization

According to Liu et al. [2005], the threshold function can be generally described by

$$T = \frac{P}{P_0} = \left[\frac{\int_{r_c}^{\infty} r^6 n(r) dr}{\int_0^{\infty} r^6 n(r) dr} \right] \left[\frac{\int_{r_c}^{\infty} r^3 n(r) dr}{\int_0^{\infty} r^3 n(r) dr} \right], \quad (2)$$

where r is the droplet radius, $n(r)$ is the cloud droplet size distributions, and r_c is the critical radius. Under the assumption that the cloud droplet size distribution is described by

$$n(r) = \frac{3N}{r_3^3} r^2 \exp \left[- \left(\frac{r}{r_3} \right)^3 \right], \quad (3)$$

the LDM threshold function was derived to be

$$T_{\text{LDM}} = \frac{1}{2} (x_c^2 + 2x_c + 2) (1 + x_c) e^{-2x_c}, \quad (4).$$

where r_3 is the mean-volume radius, and $x_c = (r_c/r_3)^3$ is the ratio of the critical mass to the mean mass of the droplet population. See *McGraw and Liu [2003, 2004]* and *Liu and McGraw [2004]* for more discussion on the critical radius/mass. The LDM threshold function holds only for a special Weibull size distribution that is described by Eq. (3) with $\varepsilon = 0.36$, and hence does not consider ε as an independent variable.

To incorporate ε , we replace Eq. (3) with the general Weibull size distribution

$$n(r) = \frac{q \Gamma^{q/3} \left(\frac{3+q}{q} \right) N}{r_3^q} r^{q-1} \exp \left[- \Gamma^{q/3} \left(\frac{3+q}{q} \right) \left(\frac{r}{r_3} \right)^q \right], \quad (5)$$

where the Weibull spectral shape parameter q is related to ε by [Liu and Daum, 2000]

$$\varepsilon = \left[\frac{2q \Gamma(2/q)}{\Gamma^2(1/q)} - 1 \right]^{1/2}. \quad (6)$$

Theoretical and observational justification for using the general Weibull droplet distribution can be found in *Liu et al. [1995]*, *Liu and Hallett [1997]*, and *Liu and Daum [2000]*. Substitution of Eq. (5) into Eqs. (2) and subsequent integration yields the

generalized threshold function

$$T_q = \frac{\Gamma\left(\frac{6+q}{q}, x_{cq}\right) \Gamma\left(\frac{3+q}{q}, x_{cq}\right)}{\Gamma\left(\frac{6+q}{q}\right) \Gamma\left(\frac{3+q}{q}\right)} = \gamma\left(\frac{6+q}{q}, x_{cq}\right) \gamma\left(\frac{3+q}{q}, x_{cq}\right), \quad (7a)$$

$$x_{cq} = \left(\frac{r_c}{r_0}\right)^q = \Gamma^{q/3} \left(\frac{3+q}{q}\right) x_c^{q/3}, \quad (7b)$$

where $\Gamma(t, z) = \int_z^\infty x^{t-1} e^{-x} dx$ is the incomplete Gamma function, and $\gamma()$ denotes the incomplete Gamma function normalized by the corresponding complete Gamma function [Press et al., 1992]. Equations (7a) and (7b) show that T_q is determined by x_c and q , and reduces to the LDM threshold function when $q = 3$.

Although T_q as given by a combination of Eqs. (6) and (7) quantifies the dependence of the threshold behavior on ε , the dependence has to be determined by repeating the procedure of calculating T_q and ε for different values of q . This is not an ideal feature for application in atmospheric models, which prefer simple relationships. However, Liu et al. [2003] showed that Eq. (6) is well approximated by

$$q \approx \varepsilon^{-1}. \quad (8)$$

It is noteworthy that Eq. (8) actually gives the exact results for $\varepsilon = 0, 1$, and ∞ , which corresponds to $q = \infty, 1$, and 0 , respectively. Substitution of Eq. (8) into Eq. (7) yields

$$T_\varepsilon = \gamma\left(\frac{6+\varepsilon^{-1}}{\varepsilon^{-1}}, \Gamma^{1/(3\varepsilon)}\left(\frac{3+\varepsilon^{-1}}{\varepsilon^{-1}}\right) x_c^{1/(3\varepsilon)}\right) \gamma\left(\frac{3+\varepsilon^{-1}}{\varepsilon^{-1}}, \Gamma^{1/(3\varepsilon)}\left(\frac{3+\varepsilon^{-1}}{\varepsilon^{-1}}\right) x_c^{1/(3\varepsilon)}\right). \quad (9)$$

The above theoretical analysis shows that the threshold function is determined by two dimensionless quantities: x_c and ε . Figure 1 shows T_q (solid lines) as a function of the

mean-to-critical mass ratio, i.e., the reciprocal of x_c at different values of ε . Also shown is T_ε (dotted lines) to compare its performance with T_q . Two points are evident from Fig. 1. First, the dependence of the threshold behavior on the mean-to-critical mass ratio gradually changes from a constant of $T = 1$ to a discontinuous δ -function as ε decreases to 0. For a given x_c , a larger ε leads to a larger value of the threshold function, and the same amount of water converted from cloudwater to rainwater requires a smaller x_c (higher liquid water content and/or lower droplet concentration) for a smaller ε . These results highlight the importance of ε in rain initiation, and are consistent with the microphysical theory that droplet collision requires relative velocities resulting from droplets of different sizes [Pruppacher and Klett, 1997]. Second, T_ε is an excellent approximation of T_q for virtually all the combinations of ε and x_c , and can be used as a substitute for T_q in practice.

3. Application of T_q to Examining ad hoc Threshold Functions

To obtain physical understanding of the commonly used ad hoc threshold functions, this section examines them by comparing to the theoretically derived generalized threshold function. Traditional ad hoc threshold functions can be generally classified as Kessler-type, Berry-type or Sundqvist-type according to their mathematical form. Briefly, the Kessler-type threshold function is a Heaviside step function [Manton and Cotton, 1977; Liou and Ou, 1989; Baker, 1993; Liu and Daum, 2004]

$$T_K = H(r_m - r_c), \quad (10)$$

where r_m and r_c denotes the driving and critical radii, respectively. The other extreme is often associated with empirical autoconversion parameterizations obtained by fitting simulations from detailed microphysical models, which generally have no threshold

functions, or implicitly assume a constant threshold function [Berry, 1968; Beheng 1994; Khairouidinov and Kogan, 2000], i.e.,

$$T_B = 1. \quad (11)$$

According to the discussion in Section 2, the Kessler-type and Berry-type threshold functions in fact only represent monodisperse and very broad cloud droplet size distributions, respectively. Furthermore, because the two extreme spectral shapes are unlikely to occur in ambient clouds [Liu and Daum, 2000], neither the Kessler-type nor the Berry-type threshold functions are generally applicable

The Sundqvist-type threshold function lies between the two extremes [Sundqvist, 1978; Del Genio et al., 1996]. However, traditional Sundqvist-type parameterizations account only for L, and are therefore unsuitable for studies of the second indirect aerosol effect. Liu et al. [2006a] has recently proposed a generalized Sundqvist-type threshold function

$$T_s = 1 - \exp[-x_c^{-\mu}], \quad (12)$$

where $\mu \geq 0$ is an empirical exponent. The generalized Sundqvist-type threshold function not only exhibits a smooth threshold behavior, but also encompasses virtually all the ad hoc threshold functions: it reduces to the traditional Sundqvist-type threshold functions proposed by Sundqvist and Del Genio when $\mu = 2$ and 4, respectively, approximately becomes the Berry-type ($T = 0.63$ not 1) when $\mu = 0$, and approaches the Kessler-type when μ approaches ∞ [See Liu et al. 2006a for detailed discussion]. Evidently, T_s is an improvement over the traditional ad hoc threshold functions that only work for some special spectral shapes of the droplet size distribution. Nevertheless, there is no physical basis for T_s , and especially, the physical meaning of μ is elusive.

Careful comparison of T_s to T_q further indicates that T_s approximately describes the overall threshold behavior of T_q , if μ is a decreasing function of ε [A trial-and-error analysis suggests that $\mu = \varepsilon^{-1}$ is not a bad approximation]. Unfortunately, T_s can accurately describe the threshold behavior only for extremely narrow droplet size distributions. For most cases, T_s is only qualitatively correct.

4. Concluding Remarks

The theoretical threshold function associated with the autoconversion process presented in *Liu et al. [2005]* is generalized to account explicitly for the effect of the relative dispersion of the cloud droplet size distribution. The generalized threshold function theoretically shows that the relative dispersion, which has been largely neglected to date, essentially controls the initial transition from cloudwater to rainwater for fixed liquid water contents and the droplet concentrations. Comparison of the generalized threshold function with existing ad hoc threshold functions shows that the threshold behavior of the autoconversion process changes from the Berry-type to the Sundqvist-type to the Kessler-type as the relative dispersion decreases, and that the empirical parameter μ in the generalized Sundqvist-type threshold function is a decreasing function of the relative dispersion, providing physical explanations for these strikingly different ad hoc threshold functions. The comparison also suggests that commonly used ad hoc threshold functions only well describe the threshold behavior for some special spectral shapes of the cloud droplet size distribution, and none of them accurately describe the threshold behavior that likely occurs in ambient clouds. These results indicate that the important role of ε in rain initiation has been unknowingly buried in the traditional practice of arbitrarily choosing ad hoc threshold functions.

It is noted in passing that although the importance of the spectral shape of the cloud droplet size distribution in rain initiation has been long recognized, the spectral shape effect has been poorly understood and quantified in atmospheric models. The new generalized threshold function is likely to improve this situation, and it will be interesting to examine the effect of the relative dispersion on modeling results using this generalized threshold function. The explicit consideration of the relative dispersion in the generalized threshold function also allows for evaluation of the effect of the relative dispersion caused by anthropogenic aerosols on rain initiation, and the second indirect aerosol effect.

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Figure Captions

Figure 1. The generalized threshold function T_q as a function of the mean-to-critical mass ratio (x_c^{-1}). The solid and dotted lines represent results calculated from T_q and its approximation T_ε , respectively. Note that most curves of T_ε overlaps with those of T_q , except those with the relative dispersion $\varepsilon = 1.76$.

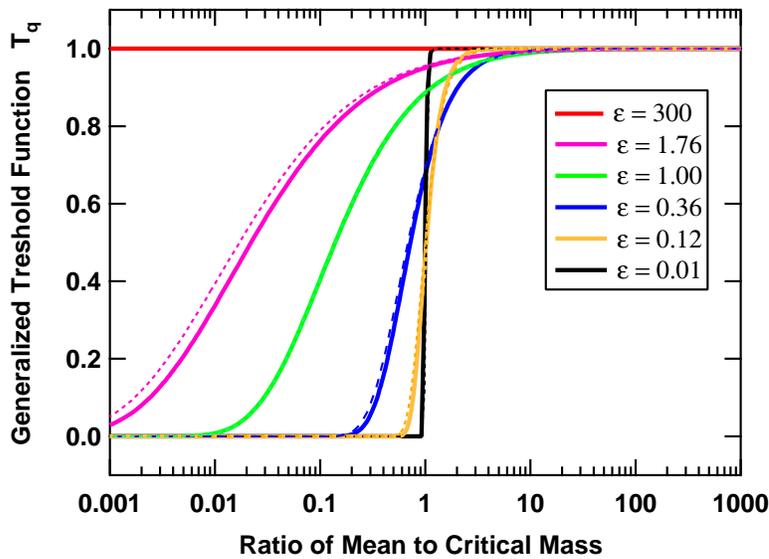


Figure 1.